Promoting Mathematical Thinking

DEPARTMENT OF EDUCATION

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Lines of development

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for the Boolean Hub

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Aims of the curriculum:

- fluency
- mathematical reasoning
- problem solving

Learning is not mechanistic

Mathematics is not mechanistic

What is a

- Master chef?
- Master craftsperson?
- Master (maestro) pianist?
- Master – anything?
- Master maths student – someone who can do the last thing they were taught quickly and correctly?

Mastery:

A well-founded accumulation of successive layers of conceptual understanding and experience.

Mathematical structures can be recognised in situations to which knowledge and methods can be applied.

This applies within mathematics and in outside contexts.

Mastery enables problem-solving and further progress in understanding.

(MT Dec 2017 p. 33; Anne Watson, Nick Andrews, Helen Drury, Jenni Ingram, Sue Lowndes, Jude Stratton, Gabriel Stylianides)

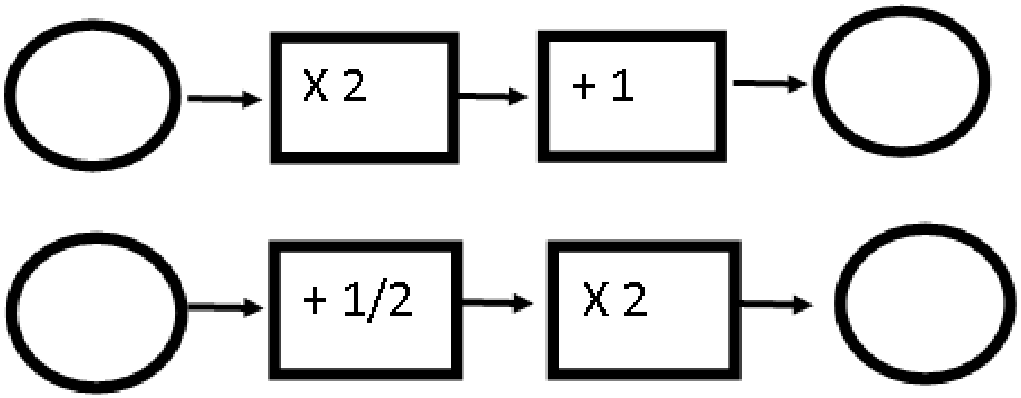
For all

Three approaches to linear relationships

Linear relationships through a designed and tested task sequence

Spot the variation

x	-1	0	1			
y				3	4	5



Linear relationships through a designed and tested task sequence

Direct/explicit instruction about what to do, but not about what to perceive

Variation of representation of one relationship

Variation of format to show equivalence

Representations: numerical and formats

Teacher prompts what has been noticed and explanations

Accessible except for physical and visual limitations

Linear graph plotting as investigation

Spot the variation

Omnigraph

Mathematical reasoning: conjecture effects on y of multiplying or adding

Problem posing: why won't it go to the other side?

Problem solving: how to put straight line graphs into the empty spaces

Mathematical reasoning: putting two ideas together to conjecture and predict shape and position of any $y=mx + c$

Fluent graph sketching

Linear graph plotting as investigation (continued…)

Direct/explicit instruction

Variation of parameters against invariant background – linear relationship

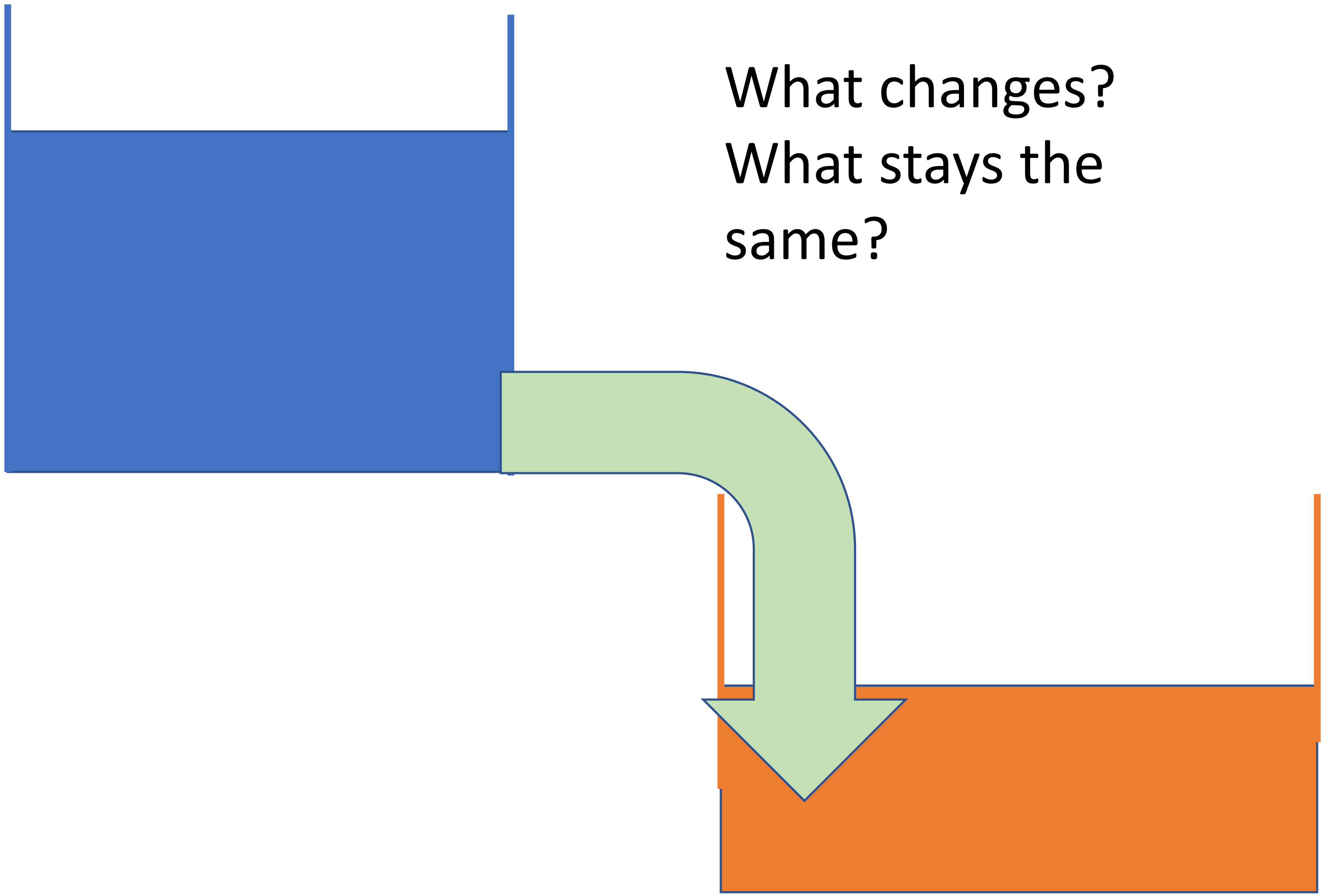
Variation as an exploration tool, providing raw material for generalisation

Representations: algebraic and graphic

Teacher questioning & problem posing: can you find a graph such that ….? (intelligent practice)

Accessible except for physical and visual limitations

Proportionality
via classification of relationships



What changes?
What stays the same?

Lesson 1

Water from one tank flows into another tank.

Q: What changes? What stays the same?

“When __ changes, __ changes/doesn’t change”

Students offer statements.

- e.g. ‘when playing time increases, studying time decreases’
- ‘When speed changes, time changes.’
- ‘When the time you sleep changes, the time you wake up changes too.’

Lesson 2

- Relationships from previous lesson are put on the board and sorted into ‘family groups’
  - Increasing/decreasing
  - Non-mathematical
  - ‘other’

Lesson 3

Teacher uses one of the phenomena:

‘As a videotape plays, time increases.’

How to investigate? Students pose a specific problem:

There is a videotape 10 m. long, rewound to the start on the left. One minute after ‘play’ has been pressed, it has advanced 1 m. to the right. If it is examined at the end of each elapsed minute, how far will the right have increased?

They suggest a table of values

Lesson 4

Solutions to several of the problems they have posed are presented and explained and discussed

Lesson 5

Focus only on the problems involving increase:

There is a pool whose capacity is 30 litres. 3 litres of water enter the pool in one minute. How will it change in one minute?

Time (mins)	1	2	3	4	5	6	7	8	9	10
Vol. of water (l)	3	6	9	12	15	18	21	24	27	30

- When time increases by 1, volume increases by 3
- If you multiply time by 3, you get volume
- If you divide volume by 3, you get time
- time and volume increase in proportion
- $x \times 3 = y$, $y \div 3 = x$
- When x increases 3 (4) times, y also increases 3 (4) times

In lesson 6 a new one is added to these:

- When x decreases to ⅓ (¼), y also decreases to ⅓ (¼)

Another problem gave the table and observations below:

Time (mins)	1	2	3	4	5	6	7	8	9	10
Vol. water (l)	1	2	3	4	5	6	7	8	9	10

- If you divide y by 1, you get x
- When x increases by 1, y also increases by 1   etc.

These were compared to the previous table and observations

Another phenomenon from lesson 4:

When one year passes, one's age increases by one.

Students had posed the specific problem:

Masako-san and her younger sister are two years apart in age. Masako-san is now 12 years old. As each year passes, how will the two girls' ages change?

Age of older	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
Age of younger	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**

When you subtract 2 from x, you get y. $x - 2 = y$.

When you add 2 to y, you get x. $y + 2 = x$.

Division and multiplication do not work, but addition and subtraction do.

Lesson 7

A student says: “y would result if you multiply x by something”.

The teacher asked: ‘Why did you say “something”?’

The student explained that the exact number was different in various problems, but in some problems even if you multiply x by something, y would not result.

Lesson 8

Four situations:

- A rectangle with a height of 8 cm has a length of x cm and an area of y cm^2.
- When you buy x apples costing 100 yen each, the price is y yen.
- When you share 5 dl of juice between your older sister and your younger brother, your sister's share is x dl and your brother's is y dl.
- The radius of a circle is x cm and the circumference is y cm.

Discuss whether these are like or not like most problems so far?

Lesson 9

From tables to graphs.

What kind of graphs?

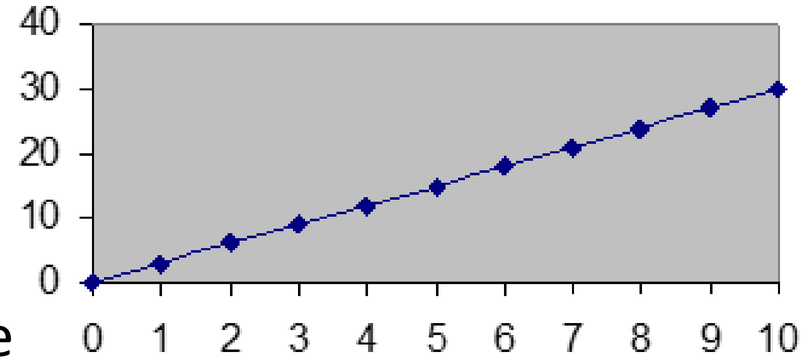
Several children suggested that line graphs would be best, and axes and scales were discussed.

Students are asked to tell the class something they could see from looking at the graph.

e.g.

"My graph went up in a diagonal straight line starting from 0."

Lesson 10



Students describe:

- continuity beyond the frame
- going up
- the meaning of the points
- going through zero (except for two situations)
- straight line
- finite situations
- not negative
- different angles with different scales
- no change in the way it increased
- can go from graph to table and back
- can find ‘new’ values (including not whole numbers)

Spot mastery

Lesson 11

Naming proportional and non-proportional relationships.

Characterising proportional relations; they say:

- When you connect x and y, you find the position of the point on the graph.
- When you go up from the position of x axis to the graph line and then go across to the y axis, you find the value of y
- The rules don’t change, the only things that change are certain numbers
- x/y is always the same

Lesson 12

There are two iron sheets of the same thickness but different shapes. One is rectangular, with sides of 6 cm and 10 cm, and weighs 120 grams. The other is an irregular shape and weighs 300 grams. Can we find the area of the second sheet?

Various ways to explain the solution were offered.

Lesson 13 (final lesson)

- Question was worked on by individuals, pairs and whole class. Move from table and graph to equation: y = something × x.
- Various versions are offered by students:

$300 \div 120 = 2.5$
$60 \times 2.5 = 150$ Answer: 150 cm^2

$300 \div x = 2 \div 1$
$300 \div 2 = 150$ Answer: 150 cm^2

Proportionality by classi... of relationships

Spot variation

Direct/explicit instruction? Discover...

Variation of types of change and situations; then a sustained situation

Students provide raw material for generalisation

Representations: verbal, tables ... graphical

Spot mastery

Teacher questioning & problem pos... the same and what is different? What is like this?

Accessible except for physical and visual limitations

Table and snakes

- **Fluency**

  making input/output tables is subordinate to thinking about the table and what it represents. Unhelpful fluency is avoided. Focus is on the functional relationship.

- **Reasoning**

  conceptual understanding of the structure of linear relationships arises from having to explain why both snakes give the same table values.

- **Solving problems**

  students have two tools now, snakes and tables, with which to represent linear relationships, and can handle variations that give the same relationship

Using a graph plotter

- **Fluency**

  recognition of algebraic and graphical forms and their characteristics: use of variation afforded by digital tools to enquire, solve and generate generalisations

- **Reasoning**

  line of enquiry triggered by conjecturing, can justify shape from algebra and vice versa

- **Solve problems**

  develop an analytical approach to graphs and algebra in which symbols have meaning

Comparison of relationships (the 13 lessons)

- **Fluency**

  conceptual understanding comes from ‘’what is? What is not?’ experiences and the gradual move towards symbolic expression and, eventually symbolic manipulation based in understanding general structure; fluency is not the end point.

- **Reasoning**

  line of enquiry is guided by teacher but with student examples and input. Students go through reasoning processes with their own familiar examples. Focus on explanation at every stage.

- **Solving problems follows posing problems**

  so building up problems precedes having to analyse given problems.

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Mathematics is not mechanistic

Mastery:

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Anne Watson

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Thinkers (ATM)

Questions and Prompts for Mathematical Thinking (ATM)

Variation: (ATM)

Key Ideas in Teaching Mathematics (OUP)